Propagation of moments and uniqueness of weak solution to Vlasov-Poisson-Fokker-Planck system

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Abstract. In this paper, we prove the uniqueness of weak solution to Vlasov-Poisson-Fokker-Planck system in $C([0,T];L^p)$, by assuming the solution has local bounded density which trends to infinite with a "reasonable" rate as t trends zero. And particularly as a corollary, we get uniqueness of weak solution with initial data f_0 satisfying $f_0|v|^2\in L^1$, which solves the uniqueness of solutions with finite energy. In addition, we prove that the moments in velocity propagate for any order higher than 2.

Keywords. Vlsov-Poisson-Fokker-Planck system, weak solution, uniqueness, propagation of moments

MSC 35Q83, 35Q84.

1 Introduction and main results

In this paper, we study three dimensional Vlsov-Poisson-Fokker-Planck system(VPFP), namely

$$\begin{cases} \partial_t f + v\cdot\partial_x f + E\cdot\partial_v f - \beta \mathrm{div}_v(vf) - \sigma\Delta_v f = 0, \\ f(0) = f_0(x,v), \\ E(t,x) = \pm\frac{1}{4\pi}\frac{x}{|x|^3} * \rho(t,x), \\ \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)dv. \end{cases} \tag{1.1}$$

Here, $\beta\geq 0$, and $\sigma\geq 0$ are given constants. In this system, function $f(t,x,v)$ is the unknown micro-density and describes the density of particles having position $x\in\mathbb{R}^3$ and velocity $v\in\mathbb{R}^3$ at time $t\geq 0$ in the phase space. The function $E(t,x)$ generated by macroscopic density $\rho(t,x)$ is called Coulombic or Gravitational force field, and can be implicitly expressed by

$$-\Delta_x V = \pm\rho = \omega\rho, \quad E = -\partial_x V.$$

The sign $\omega = 1$ corresponds to the Coulombic interaction whereas the sign $\omega = -1$ described the gravitational interaction between the particles. Finally, the term $-\beta\mathrm{div}_v(vf)$ corresponds to the friction effects in the fluid, and the term $-\Delta_v f$ describes grazing collisions between particles when colliding.

Before gonging to details, we give a summary of the researches on Vlasov-Poisson and VPFP system. In the classical Vlasov-Poisson case, namely $\beta = 0, \sigma = 0$, P.L. Lions and B. Perthame [3] proved that $f_0 \in L^1\bigcap L^\infty$, $|v|^m f_0\in L^1$, for some $m>3$, implies the existence of a solution $f(t,x,v)$ satisfying $|v|^m f\in L^1$ for all t. Uniqueness was also considered in [3], roughly speaking

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they proved that a Lipschitz condition for initial data could lead to the uniqueness of weak solution. For smooth data, K. Paffelmoser [4] proved the global existence and uniqueness of the smooth solution with compact support to Vlaosv-Poisson system. Gerhard Rein [5] is a nice review for classical solutions to Vlasov-Poisson equation. Based on optimal transportation, Gregoire Loeper [6] proved uniqueness by assuming density $\rho(t) \in L^\infty$.

In the VPFP case, $\sigma > 0$. F. Bouchut [1] proved the existence of weak solution to Vlsov-Poisson-Fokker-Planck system when the initial data satisfies $f_0 \in L^1 \bigcap L^\infty$, and $|v|^m f_0 \in L^1$, for some $m > 6$. Furthermore, F. Bouchut [1] proved uniqueness of mild solutions in the space $f(t,x,v) \in C([0,T]; L^1)$, $E(t,x) \in L^\infty$. F. Bouchut's proof relied on a similar technique introduced in P.L. Lions and B. Perthame [3], but the regularizing effect of the diffusion term $-\Delta_v f$ was also essential there. F. Bouchut [2] discovered the smoothing effect which says $|v|^2 f_0 \in L^1$ and $f_0 \in L^1 \bigcap L^\infty$ can lead to $E(t,x) \in L^\infty_{loc}((0,T]; \mathbb{R}^6)$, more precisely, $\|E(t,x)\|_{L^\infty} \le t^{-\gamma}$ for $t > 0$ and some $\gamma > 0$. F. Castella [7] built solutions to this system having infinite kinetic energy, namely $|v|^\alpha f_0 \in L^1$, $\alpha \in (0,2)$. Moereover, [7] also proved the smoothing effect mentioned just above. A. Carpio [8] studied the long time behavior of VPFP, and proved the solution scatters to the linear VPFP system.

From now on, we focus on uniqueness. As we mentioned, in VPFP case, The uniqueness for solutions with bounded $E(t,x)$ has been established by F. Bouchut [1]. And in Vlaosv-Poisson case, namely $\beta = 0$, $\sigma = 0$, Gregoire Loeper [6] proved uniqueness for solutions with $\rho(t) \in L^\infty$.

In this paper, for VFPP eqaution, we try to remove the bounded-ness assumptions in both [1] and [6]. In fact, we will prove uniqueness for $\|t^\gamma \rho(t,x)\|_{L^\infty} < \infty$, $\|t^\mu E(t,x)\|_{L^\infty} < \infty$, for some $\gamma > 0, \mu > 0$. And due to the smooth effect mentioned above, the local bounded assumptions of ρ and E we used here are naturally satisfied. Besides, we emphasize that it is known that when the initial data $f_0|v|^m \in L^1$, for some $m > 6$, then E is bounded, thus in this case result in [1] is enough to give a uniqueness theory; however, the uniqueness of solution with initial data with moments in v less than 6 remains an open problem; the most important case of uniqueness is the 2-order moment case, which lies in the energy level, and our result solves it.

Besides uniqueness, the other interesting problem is whether the moment propagates or not. In the Vlasov-Poisson case, P.L. Lions and B. Perthame [3] proved the moments higher than three propagate; Christophe Pallard [9] showed any moment higher than two-order preserves. In VPFP case, F. Bouchut [1] proved moments bigger than six propagates. Since 2-order is the energy level, which of course preserves, the remaining problem is does moment from two to six propagate? In this paper, we give a positive answer to this problem.

One of the essential ingredients of our proof is the approximating proposition, namely Proposition 2.9, which aims to relax the conditions of test functions in the definition of weak solution. It says that a rough solution to a degenerate parabolic linear equation with singular coefficients can be approximated by Schwartz solutions in L^p space. The proof relies on the duality arguments, which reduce the problem to the uniqueness of the dual linear equation. While using duality method, we apply Fourier truncation technique to overcome some difficulties originating from the fact that we are dealing with rough solutions for which integration by parts or other flexibility fails.

After establishing the approximating proposition, although we can take any rough function, if we like, to be the test function, but when we deal with two weak solutions, this proposition is not enough to provide us the same convenience. That is why Proposition 3.3 is needed. This proposition says weak solution in fact has some regularity in v variable, which is not too surprising, since when $\sigma \neq 0$, VPFP has a diffusion term. Combining Proposition 2.9 and Proposition 3.3, we successfully low down the regularity requirements of test functions and can choose them freely.

Another essential ingredient in our proof is the choosing of a proper norm to obtain a Gronwall inequality. We first reduce the uniqueness of weak solution to the being zero of the two force fields generated by the two solutions which we suppose to exit. Then take an appropriate test function to obtain a Gronwall inequality, by which we can end the proof.

In the paper, we only consider $\sigma > 0$. The definition of weak solution is the follows.

Definition 1.1. A function $f(t,x,v)$ is called the weak solution to VPFP, if for all $\varphi(x,v,t) \in C^1([0,T]; S(\mathbb{R}^3 \times \mathbb{R}^3))$ with $\varphi(T,x,v) = 0$, it holds that

$$\int_{\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3} f(\partial_t\varphi + v\cdot\partial_x\varphi + E\cdot\partial_v\varphi - \beta v\cdot\partial_v\varphi + \sigma\Delta_v\varphi)dtdxdv + \int_{\mathbb{R}^3\times\mathbb{R}^3} f_0\varphi(0)dxdv = 0. \tag{1.2}$$

where S is the Schwartz class.

The main theorems we obtain are the followings.

On propagation of moments, we will prove

Theorem 1.2. *If the initial data f_0 satisfies $f_0(1+|v|^k) \in L^1$, $f_0 \in L^\infty$, for some $k \geq 2$, then there exists a weak solution to VPFP system satisfying $f(t)(1+|v|^k) \in L^\infty([0,T]; L^1)$.*

For uniqueness, we have

Theorem 1.3. *If $f_0 \in L^1 \bigcap L^\infty$, $f_0|v|^2 \in L^1$, then there exists one and only one weak solution to VPFP system satisfying $f(t,x,v) \in C([0,T]; L^p)$ for any $p \in [1,\infty)$, $\|E(t,x)\|_r \leq C$, for $r \in (3/2, 15/4)$, $\|E(t,x)\|_{L^\infty} \leq C(T)t^{-6/5}$, and $\|\rho(t,x)\|_{L^q} \leq C(T)t^{-27/10+9/2q}$, for each $q \in [5/3, \infty]$.*

The existence part follows from Theorem 1 in F. Bouchut [2]. And we mention that, in fact, F. Bouchut [1] proved uniqueness for mild solution in some sense, which is stronger than the weak solution we consider here. Finally, we point out that our proof of Theorem 1.3 in fact can be modified to prove uniqueness for weak solution with moment higher than k_0, where k_0 is some number strictly less than two.

This paper is arranged as follows. In section 2, we carefully study the linear problem. In section 3, we reduce the uniqueness problem to the being zero of the difference of the two force fields. In section 4, we deduce Theorem 1.3. In section 5, we prove Theorem 1.2.

All the constants are denoted by C or c_j , $j \in N^+$. They will change from line to line. And c_j are absolute constants.

We use E_f to emphasize the force field is generated by micro-density f.

The Fourier transform is denoted by F. Now we introduce the Fourier truncation operator and modifying operator as follows. Let Φ be an infinitely smooth function compactly supported in the unit ball in $\mathbb{R}^6$, and equals 1 in the ball with radius $\frac{1}{2}$. Then define

$$\Psi(\xi,\zeta) = \Phi(\frac{\xi}{2^n})\Phi(\frac{\zeta}{2^m}),$$

and Fourier truncation $P_{n,m} = F^{-1}_{\xi,\zeta}\Phi(\frac{\xi}{2^n})\Phi(\frac{\zeta}{2^m})F_{x,v}$, and smooth modifying operator $Q_k\eta = \eta *_{[0,T]} \alpha_k$, where $\alpha_k(t)$ is the standard modifier in $\mathbb{R}$.

2 Linear problem

Proposition 2.1. *Consider the following equation*

$$\partial_t \psi - v\cdot\partial_x\psi + \beta v\cdot\partial_v\psi - \sigma\Delta_v\psi = 0,$$

there exists a fundamental solution $G(x,v,t;x_0,v_0)$:

$$\begin{aligned}G =&\frac{e^{3\beta t}}{(2\pi)^6}\left(\frac{\pi}{\sigma\sqrt{D}}\right)^3 \exp\{\frac{-1}{4\sigma D}[\frac{1}{2\beta}(e^{2\beta t}-1)|\bar{x}|^2\\ &-\left(\frac{1}{\beta^2}(1-e^{\beta t})-\frac{1}{2\beta^2}(1-e^{2\beta t})\right)\langle\bar{x},\bar{v}\rangle\\ &+\left(\frac{t}{\beta^2}+\frac{2}{\beta^3}(1-e^{\beta t})-\frac{1}{2\beta^3}(1-e^{2\beta t})\right)|\bar{v}|^2]\},\end{aligned}$$

where $D=\frac{\beta(e^{2\beta t}-1)t-2(1-e^{\beta t})^2}{2\beta^4}$, and $\bar{x}=x-(x_0+\frac{v_0}{\beta}(e^{\beta t}-1))$, and $\bar{v}=v-v_0e^{\beta t}$.

The calculations are standard, thus we omit the details.

Lemma 2.2. *Define*

$$Tf(x,v,t)=\int_0^t\int_{R^6} G(x,v,t-\tau;x_0,v_0)f(x_0,v_0,\tau)dx_0dv_0d\tau,$$

then

$$\|Tf(t)\|_{L^p(\mathbb{R}^6)} \le C\int_0^t \|f(\tau)\|_{L^p(\mathbb{R}^6)}d\tau \tag{2.3}$$

$$\|\partial_v Tf(t)\|_{L^p(\mathbb{R}^6)} \le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|f(\tau)\|_{L^p(\mathbb{R}^6)}d\tau. \tag{2.4}$$

Proof By Schur's Lemma, (2.3) is a consequence of the following elementary property.

$$\int_{\mathbb{R}^6} G(x,v,t;x_0,v_0)dxdv \le c_1; \int_{\mathbb{R}^6} G(x,v,t;x_0,v_0)dx_0dv_0 \le c_2. \tag{2.5}$$

(2.4) is a result of the following inequality and (2.3):

$$|\partial_v G(x,v,t-\tau;x_0,v_0)| \le C\frac{1}{(t-\tau)^{1/2}}G(\frac{x}{2},\frac{v}{2},t-\tau;\frac{x_0}{2},\frac{v_0}{2}). \tag{2.6}$$

Proposition 2.3. *Consider the following linear equation*

$$\partial_t \psi - v\cdot\partial_x\psi + \beta v\cdot\partial_v\psi - \sigma\Delta_v\psi - E\cdot\partial_v\psi = 0, \tag{2.7}$$

assume $\|E\|_{L^\infty([0,T]\times R^3)} \le Ct^{6/5}$, then for small T, there exists a fundamental solution Γ which solves

$$\begin{aligned}&\Gamma(x,v,t;x_0,v_0,\tau) = G(x,v,t-\tau;x_0,v_0) \\ &+\int_\tau^t\int_{\mathbb{R}^6}\partial_{v_1}G(x,v,t-s;x_1,v_1)E(s,x_1)\Gamma(x_1,v_1,s;x_0,v_0,\tau)dx_1dv_1ds\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&= -\int_\tau^t \int_{\mathbb{R}^6} G(x,v,t-s;x_1,v_1)E(s,x_1)\partial_{v_1}\Gamma(x_1,v_1,s;x_0,v_0,\tau)dx_1dv_1ds \\
&+ G(x,v,t-\tau;x_0,v_0).
\end{aligned}
$$

And

$$
\begin{aligned}
|\Gamma(x,v,t;x_0,v_0,\tau)| &\le C(t-\tau)^{1/5}G(\frac{x}{2},\frac{v}{2},t-\tau;\frac{x_0}{2},\frac{v_0}{2}) \\
|\partial_v\Gamma(x,v,t;x_0,v_0,\tau)| &\le \frac{C}{(t-\tau)^{\frac{7}{10}}}G(\frac{x}{2},\frac{v}{2},t-\tau;\frac{x_0}{2},\frac{v_0}{2}).
\end{aligned}
$$

Proof Define function space

$$
X = \left\{ g \,\middle|\, \begin{array}{l} g(x,v,t;x_0,v_0,\tau) \le C(t-\tau)^{-1/5}G(x/2,v/2,t-\tau,x_0/2,v_0/2); \\ \partial_v g(x,v,t;x_0,v_0,\tau) \le C(t-\tau)^{-\frac{7}{10}}G(x/2,v/2,t-\tau,x_0/2,v_0/2). \end{array} \right\},
$$

with norm

$$
\begin{aligned}
\|g\|_X &= \left\| \frac{(t-\tau)^{1/5}g(x,v,t;x_0,v_0,\tau)}{G(x/2,v/2,t-\tau,x_0/2,v_0/2)} \right\|_{L^\infty([0,T]^2\times\mathbb{R}^{12})} \\
&\quad + \left\| \frac{(t-\tau)^{7/10}\partial_v g(x,v,t;x_0,v_0,\tau)}{G(x/2,v/2,t-\tau,x_0/2,v_0/2)} \right\|_{L^\infty([0,T]^2\times\mathbb{R}^{12})} \\
&\triangleq \|g\|_{X_1} + \|g\|_{X_2}
\end{aligned}
$$

Then define

$$
E = \{g : \|g\|_X \le R\}.
$$

We will apply fixed-point principle to equation (2.8) in E.
Let

$$
\begin{aligned}
\Im g(x,v,t;x_0,v_0,\tau) &= \int_\tau^t \int_{\mathbb{R}^6} \partial_{v_1}G(x,v,t-s;x_1,v_1)E(s,x_1)g(x_1,v_1,s;x_0,v_0,\tau)dx_1dv_1ds \\
&\quad + G(x,v,t-\tau;x_0,v_0)
\end{aligned}
$$

It is easy to see

$$
\begin{aligned}
&\|\Im g_1 - \Im g_2\|_{X_1} \\
&\le \left\| \frac{\int_\tau^t \int_{\mathbb{R}^6} \partial_{v_1}G(x,v,t-s;x_1,v_1)E(s,x_1)G(x_1/2,v_1/2,s-\tau,x_0/2,v_0/2)dx_1dv_1ds}{(t-\tau)^{-1/5}(s-\tau)^{1/5}G(x/2,v/2,t-\tau,x_0/2,v_0/2)} \right\|_\infty \\
&\quad \times C\|g_1-g_2\|_{X_1} \\
&\le C\left\| \int_\tau^t \int_{\mathbb{R}^6} \frac{G(x/2,v/2,t-s;x_1/2,v_1/2)G(x_1/2,v_1/2,s-\tau,x_0/2,v_0/2)}{(t-\tau)^{-1/5}(s-\tau)^{1/5}s^{6/5}(t-s)^{1/2}G(x/2,v/2,t-\tau,x_0/2,v_0/2)} \right\|_\infty \\
&\quad \times C\|g_1-g_2\|_{X_1} \\
&\le C\|g_1-g_2\|_{X_1} \sup \int_\tau^t \frac{(t-\tau)^{1/5}}{(t-s)^{1/2}s^{6/5}(s-\tau)^{1/5}}ds
\end{aligned}
$$

$$\leq CT^{1/2}\|g_1-g_2\|_{X_1}$$

where we have used (2.6) and the following obvious equality

$$\int_{R^6} G(x/2,v/2,t-s;x_1/2,v_1/2)G(x_1/2,v_1/2,s-\tau,x_0/2,v_0/2)dx_1dv_1 \\ = G(x/2,v/2,t-\tau;x_0/2,v_0/2).$$

And similarly, it holds that

$$\begin{aligned}
&\|\Im g_1-\Im g_2\|_{X_2}\\
&\leq \left\|\int_\tau^t \frac{\int_{\mathbb{R}^6}\partial_v\partial_{v_1}G(x,v,t-s;x_1,v_1)E(\tau,x_1)(g_1-g_2)(x_1,v_1,s;x,v,\tau)dx_1dv_1}{(t-\tau)^{-7/10}G(x/2,v/2,t-\tau,x_0/2,v_0/2)}ds\right\|_\infty\\
&\leq \left\|\int_\tau^t \frac{\int_{\mathbb{R}^6}\partial_v G(x,v,t-s;x_1,v_1)E(s,x_1)\partial_{v_1}(g_1-g_2)(x_1,v_1,s;x,v,\tau)dx_1dv_1}{(t-\tau)^{-7/10}G(x/2,v/2,t-\tau,x_0/2,v_0/2)}ds\right\|_\infty\\
&\leq \left\|\int_\tau^t \frac{\int_{\mathbb{R}^6}G(x/2,v/2,t-s;x_1/2,v_1/2)G(x_1/2,v_1/2,s-\tau,x_0/2,v_0/2)}{s^{-6/5}(t-\tau)^{-7/10}(s-\tau)^{7/10}(t-s)^{1/2}G(x/2,v/2,t-\tau,x_0/2,v_0/2)}\right\|_\infty\\
&\quad\times C\|g_1-g_2\|_{X_2}\\
&\leq C\|g_1-g_2\|_{X_2}\sup\int_\tau^t \frac{(t-\tau)^{\frac{7}{10}}}{(s-\tau)^{7/10}(t-s)^{1/2}s^{6/5}}ds\\
&\leq CT^{1/2}\|g_1-g_2\|_{X_2}
\end{aligned}$$

Follow the same arguments above, it is direct that

$$\|\Im g\|_X \leq c_1 + CT^{1/2}\|g\|_X.$$

where c_1 is a number.
Thus we have proved $\Im$ is a contraction map in E, provided R is large enough, and T is small enough. Therefore, not only do we have proved the existence of solution to (2.8), but also we get the two estimates in Proposition 2.3. It is obvious Γ is just the fundamental solution considering the two estimates we obtain of Γ and the equation (2.8).

Remark 2.4. In Proposition 2.3, we only get the existence of Γ in a small interval $[0,T_1]$. But, we find in the proof above, T_1 can be determined as an absolute constant. It is easy to see if Theorem 1.3 holds in $[0,T_1]$, for some number T_1, then Theorem 1.3 holds for any lifespan $[0,T]$. Thus the local existence of fundamental solution is enough for our purpose.

Remark 2.5. By similar arguments, we can prove

$$|\nabla_{v_0}\Gamma(x,v,t;x_0,v_0,\tau)| \leq \frac{C}{(t-\tau)^{7/10}}G(x/2,v/2,t;x_0/2,v_0/2),$$

in fact it follows from

$$|\nabla_{v_0}G(x,v,t;x_0,v_0,\tau)| \leq \frac{C}{(t-\tau)^{1/2}}G(x/2,v/2,t;x_0/2,v_0/2).$$

Proposition 2.6. *Suppose $\|E(t,x)\|_\infty \le Ct^{-6/5}$. Consider*

$$\partial_t\psi - v\cdot\partial_x\psi + \beta v\cdot\partial_v\psi - \sigma\Delta_v\psi - E\cdot\partial_v\psi = f,$$
$$\psi(0) = 0. \tag{2.9}$$

For any $f \in L^\infty([0,T];L^1\bigcap L^p)$, $p \ge 2$, there exits a solution $\psi \in C([0,T];L^1\bigcap L^p)$ to (2.9), and $\|\partial_v\psi(t)\|_{L^r} \le \int_0^t (t-\tau)^{-7/10}\|f(\tau)\|_{L^r}$, for each $1 \le r \le p$. Furthermore, smooth solutions are unique.

Proof Define

$$T_2f(x,v,t) = \int_0^t \int_{\mathbb{R}^6} \Gamma(x,v,t;x_0,v_0,\tau)f(x_0,v_0,\tau)dx_0dv_0d\tau.$$

Then $T_2f(x,v,t)$ is just a solution to (2.9) according to the definition of fundamental solution and Duhamel's principle.
From Proposition 2.3 and Proposition 2.2, we deduce

$$\|T_2f(t)\|_{L^p(R^6)} \le C\int_0^t (t-\tau)^{-1/5}\|f(\tau)\|_{L^p(R^6)}d\tau \tag{2.10}$$
$$\|\partial_vT_2f(t)\|_{L^p(\mathbb{R}^6)} \le C\int_0^t (t-\tau)^{-\frac{7}{10}}\|f(\tau)\|_{L^p(\mathbb{R}^6)}d\tau$$

Thus the remaining part is the uniqueness.
Suppose, there are two smooth solutions to (2.9), and their difference is η, then

$$\begin{cases} \partial_t\eta - v\cdot\partial_x\eta + \beta v\cdot\partial_v\eta - \sigma\Delta_v\eta - E\cdot\partial_v\eta = 0, \\ \eta(0) = 0 \end{cases}$$

Taking inner product with η, using integration by parts,

$$\frac{1}{2}\frac{1}{dt}\|\eta\|_2^2 - \frac{3\beta}{2}\|\eta\|_2^2 + \sigma\|\partial_v\eta\|_2^2 = 0.$$

As a consequence of Gronwall inequality, $\eta = 0$.

Remark 2.7. From Remark 2.5, if we define

$$T_3g = \int_{\mathbb{R}^6} \Gamma(x,v,t;x_0,v_0,\tau)g(x,v)dxdv,$$

then for $p \in [1,\infty]$,

$$\|\partial_{v_0}T_3g\|_{L^\infty([0,T];L^p)} \le C\|g\|_p;$$
$$\|T_3g\|_{L^\infty([0,T];L^p)} \le C\|g\|_p.$$

Remark 2.8. From the transformation $t \to T-t$, all the results in this section can be extended parallel to the following equation,

$$\begin{cases} \partial_t\varphi + v\cdot\partial_x\varphi + E\cdot\partial_x\varphi - \beta v\cdot\partial_x\varphi + \sigma\Delta_v\varphi = f \\ \varphi(T) = 0 \end{cases} \tag{2.11}$$

The corresponding fundamental solution will be denoted by Γ_1.
And consider

$$-\partial_t\varphi - v\cdot\partial_x\varphi - E\cdot\partial_x\varphi + \beta v\cdot\partial_x\varphi + \sigma\Delta_v\varphi = f. \tag{2.12}$$

Similar discussions indicates the existence of fundamental solution, and we denote it as Γ_2.

Proposition 2.9. *Assume $E(t,x)$ satisfies conditions in Theorem 1.3, then for any $f \in L^\infty([0,T];L^p)$, for any $p \in [1,\infty]$, there exits a unique solution to equation (2.11), and a sequence of functions $\varphi_n \in C^\infty([0,T];S(\mathbb{R}^6)$ such that*

$$\partial_t\varphi_n + v\cdot\partial_x\varphi_n + E\cdot\partial_v\varphi_n - \beta v\cdot\partial_x\varphi_n + \sigma\Delta_v\varphi_n \to f \text{ in } L^2([0,T]\times\mathbb{R}^6),$$
$$\varphi_n \to \varphi \quad \text{in } C([0,T];L^p(\mathbb{R}^6)) \text{ for } p\in[2,\infty).$$

Proof The existence of solution to (2.11) is included in Remark 2.8. For the existence of φ_n, we split the proof into three steps.

Step1. Define Hilbert space $H = L^2([0,T]\times\mathbb{R}^6)$, and a linear operator L with domain

$$D(L) = \left\{\varphi \in C^\infty([0,T];S(\mathbb{R}^6))\,|\varphi(T)=0\right\},$$

where S is the Schwartz class, and

$$L\varphi = \partial_t\varphi + v\cdot\partial_x\varphi + E\cdot\partial_x\varphi - \beta v\cdot\partial_v\varphi + \sigma\Delta_v\varphi.$$

Then
$$\left\{\varphi\in C^\infty([0,T];C_c^\infty(\mathbb{R}^6))\,|\varphi(0)=0\right\}\subseteq D(L^*),$$
thus L^* is densely defined, and hence L is closeable. Define the closure of L as $\mathbb{L}$, with domain $D(\mathbb{L})$, then
$$\mathbb{L}^*\varphi = -\partial_t\varphi - v\cdot\partial_x\varphi - E\cdot\partial_x\varphi + \beta v\cdot\partial_v\varphi + \sigma\Delta_v\varphi.$$
Now we introduce an auxiliary space Y' defined as the dual space of

$$Y = \left\{ g \,\middle|\, (1+|v|)(|g|+|\nabla g|+|\Delta_v g|)\in L^2, \int_{\mathbb{R}^3}|\partial_v g|dv \in L^{2r/(r-2)}\right\}.$$

Then from $\mathbb{L}^*\varphi = f$, we know

$$\partial_t\varphi = -f - v\cdot\partial_x\varphi - E\cdot\partial_v\varphi + \beta v\cdot\partial_v\varphi + \sigma\Delta_v\varphi,$$

thus since $\|E\|_r$ is bounded in $[0,T]$, it follows

$$\begin{aligned}
&\int_0^T \|\partial_t\varphi\|_{Y'}dt\\
&= \int_0^T \sup_{\|g\|_Y=1}\int_{\mathbb{R}^6} g(-f - v\cdot\partial_x\varphi - E\cdot\partial_v\varphi + \beta v\cdot\partial_v\varphi + \sigma\Delta_v\varphi)dtdxdv\\
&\le C\int_0^T \sup_{\|g\|_Y=1} \|(1+|v|)(|g|+|\nabla g|+|\Delta_v g|)\|_2\|\varphi\|_2 + \|E\cdot\partial_v g\|_2\|\varphi\|_2 + \|g\|_2\|f\|_2 dt\\
&\le C\int_0^T \sup_{\|g\|_Y=1} \|(1+|v|)(|g|+|\nabla g|+|\Delta_v g|)\|_2\|\varphi\|_2 + \|E\|_r\left\|\int_{\mathbb{R}^3}|\partial_v g|\,dv\right\|_{2r/(r-2)}\|\varphi\|_2\\
&\quad + \|g\|_2\|f\|_2 dt\\
&\le C\|\varphi\|_{L^2([0,T]\times R^6)} + C\|f\|_{L^2([0,T]\times R^6)}.
\end{aligned}$$

Therefore φ has a continuous correction in Y', without loss of generalization, we can assume $\varphi\in C([0,T];Y')$.
Now we claimt if $\eta\in D(\mathbb{L}^*)$, then $\eta\in C([0,T];Y')$, and $\eta(0)=0$. The proof see Appendix A.

Step 2. We claim $\ker(\mathbb{L}^*) = 0$. Namely, if $\mathbb{L}^*\eta = 0$, then $\eta = 0$.
The proof relies on Fourier truncation, uniqueness for smooth solution of (2.12), and fundamental solution.
Apply Q_k, $P_{n,m}$ to $\mathbb{L}^*\eta = 0$, we find

$$\mathbb{L}^* Q_k P_{n,m}\eta = -\partial_x F^{-1}\partial_\zeta \Psi F Q_k\eta + \beta\partial_v F^{-1}\partial_\zeta\Psi F Q_k\eta + Q_k P_{n,m} E\cdot\partial_v\eta - E\cdot\partial_v Q_k P_{n,m}\eta.$$

For each $g \in C_c^\infty(\mathbb{R}^6)$, define

$$\int_{R^6} \Gamma_2(x,v,t;x_0,v_0,\tau) g(x,v)dxdv = \tilde{g}(x_0,v_0,t,\tau).$$

Then from Remark 2.7, we find $\|\tilde{g}\|_{L^2} \le C$, $\|\partial_{v_0}\tilde{g}\|_{L^\infty} \le C$.
Now from the uniqueness of smooth solution of (2.12) and the representation of $Q_k P_{n,m}\eta$ via fundamental solution, we have

$$\begin{aligned}
\langle Q_k P_{n,m}\eta(t), g\rangle - &\int_{\mathbb{R}^6}\int_{\mathbb{R}^6} g(x,v)\Gamma_2(x,v,t;x_0,v_0,0)\left(P_{n,m}Q_k\eta\right)(0,x_0,v_0)\\
&= \int_{\mathbb{R}^6} g(x,v)dxdv\int_0^t\int_{\mathbb{R}^6}\Gamma_2(x,v,t;x_0,v_0,\tau)(-\partial_{x_0}F^{-1}\partial_\zeta\Psi F Q_k\eta +\\
&\quad \beta\partial_{v_0}F^{-1}\partial_\zeta\Psi F Q_k\eta + Q_k P_{n,m}E\cdot\partial_{v_0}\eta - E\cdot\partial_{v_0}P_{n,m}Q_k\eta)d\tau dx_0 dv_0\\
&= \int_0^t\int_{\mathbb{R}^6}\tilde{g}(x_0,v_0,t,\tau)(-\partial_{x_0}F^{-1}\partial_\zeta\Psi F\eta + \beta\partial_{v_0}F^{-1}\partial_\zeta\Psi F\eta +\\
&\quad Q_k P_{n,m}E\cdot\partial_{v_0}\eta - E\cdot\partial_{v_0}P_{n,m}Q_k\eta)d\tau dx_0 dv_0.
\end{aligned}$$

Let $k\to\infty$, from Remark 2.7, Bernstein's inequality, and $\eta(0)=0$, we obtain

$$\begin{aligned}
&|\langle P_{n,m}\eta(t), g\rangle|\\
&\le \int_0^t \frac{C2^n}{2^m}\|\tilde{g}\|_2\|\eta\|_2 + \|E\|_2\|\eta\|_2\left\|\int_{\mathbb{R}^3} P_{n,m}\partial_{v_0}\tilde{g} - \partial_{v_0}\tilde{g}\right\|_\infty\\
&\quad + \|P_{n,m}\eta - \eta\|_2\left\|\int_{\mathbb{R}^3}\partial_{v_0}\tilde{g}\right\|_\infty \|E\|_2 dt\\
&\le C\|\eta\|_{L^2([0,T];L^2)}\left(\|\tilde{g}\|_{L^\infty([0,T];L^2)}\frac{2^n}{2^m} + \left\|\int_{\mathbb{R}^3} P_{n,m}\partial_{v_0}\tilde{g} - \partial_{v_0}\tilde{g}dv\right\|_{L^\infty([0,T];L^\infty)}\right)\\
&\quad + \|P_{n,m}\eta - \eta\|_{L^2([0,T];L^2)}\|\partial_{v_0}\tilde{g}\|_{L^\infty([0,T];L^\infty)}.
\end{aligned}$$

Let $m\to\infty$, then $n\to\infty$, it follows

$$\lim_{n,m\to\infty}\langle P_{n,m}\eta(t), g\rangle = 0.$$

Since $P_{n,m}\eta(t)\to\eta(t)$ in $C([0,T];Y')$, then $\eta(t)=0$.
Step 3. From Step2, we know $(\mathbb{L}^*)^{-1}$ is a linear bounded operator from $R(\mathbb{L}^*)$ to $D(\mathbb{L}^*)$. Then $(\mathbb{L}^*)^{-1}$ can be extended to $\overline{R(\mathbb{L}^*)}$ to $D(\mathbb{L}^*)$, then extended to $S: H\to H$.
We say $\varphi\in H$ is a weak solution to (2.11), if for any $y\in D(\mathbb{L}^*)$, it holds that

$$\langle f, y\rangle = \langle\varphi, L^* y\rangle. \tag{2.13}$$

It is obvious $\varphi = S^* f$ satisfies (2.13), thus we have given the other proof of the existence to (2.11), but this approach can reach further than existence. Another view to regard (2.13) is

that φ is in the domain of $\mathbb{L}^{**}$. Since $\mathbb{L}^*$ is closed, then $\mathbb{L}^{**} = \mathbb{L}$. Therefore $\varphi \in D(\mathbb{L})$. Since $D(L)$ is the core of $D(\mathbb{L})$, form the definition of core, we find there exits a sequence of functions $\varphi_n \in C^\infty([0,T]; S(\mathbb{R}^6))$ such that

$$\begin{aligned}\partial_t\varphi_n + v\cdot\partial_x\varphi_n + &E\cdot\partial_x\varphi_n - \beta v\cdot\partial_x\varphi_n + \sigma\Delta_v\varphi_n \to f\\ &\text{in } L^2([0,T]\times\mathbb{R}^6),\\ &\varphi_n \to \varphi \text{ in } L^2([0,T]\times\mathbb{R}^6).\end{aligned}$$

The proof of uniqueness to the solution of (2.11) in $C([0,T];Y')$ is all the sane as the proof of $ker\mathbb{L}^* = 0$. Thus all the discussions of the solution built in Remark 2.8 can be moved here. Then from (2.10), we have $\varphi_n \to \varphi$ in $C([0,T];L^2(\mathbb{R}^6))$, and by interpolation, $\varphi_n \to \varphi$ in $C([0,T];L^p(\mathbb{R}^6))$ for any $p \in [2,\infty)$.

Remark 2.10. Proposition 3 enables us to take the solution to (2.11) as a test function in Definition 1.1, by a limit argument.

3 Reduce the problem to $E_w = 0$

From now on, we suppose there are two solutions f_1 and f_2 satisfying the conditions in Theorem 1.3. And define $w = f_1 - f_2$.

Lemma 3.1.

$$\|\int_{\mathbb{R}^3} E(y)\frac{y-x}{|x-y|^3}dy\|_\infty \le C.$$

Proof It is a result of $\|E\|_p \le C$, for $p \in (\frac{3}{2}, \frac{15}{4})$. In fact,

$$\begin{aligned}&\left|\int_{R^3} E(y)\frac{y-x}{|x-y|^3}dy\right|\\ &\le \left|\int_{|x-y|\ge1} E(y)\frac{y-x}{|x-y|^3}dy\right| + \left|\int_{|x-y|\le1} E(y)\frac{y-x}{|x-y|^3}dy\right|\\ &\le \|E(y)\|_r\left\|\frac{1_{|x-y|\ge1}}{|x-y|^2}\right\|_{r'} + \|E(y)\|_p\left\|\frac{1_{|x-y|\le1}}{|x-y|^2}\right\|_{p'}\\ &\le C\|E(y)\|_r + C\|E(y)\|_p,\end{aligned}$$

where $1 < r < 3$ and $p > 3$.

Define the non-negative smooth function δ satisfying $\delta(t) = 0$ for $|t| \ge 1$, and $\int_{\mathbb{R}} \delta(t)dt = 1$. Let $\delta_{s,\theta}(t) = \frac{1}{\theta}\delta(\frac{t-s}{\theta})$.

Lemma 3.2. *If $f(t) \in C([0,T])$, then*

$$\lim_{\theta\to0}\int_0^T f(t)\delta_{s,\theta}(t)dt = f(s)$$

Proof The proof is direct and omitted. □

Proposition 3.3. *If f is a weak solution in Definition 1.1, $f_0 \in L^2$, E satisfies the conditions in Theorem 1.3, then we actually have $\partial_v f \in L^{p'}([0,T];L^2)$, where $p = \frac{10}{3}^+$.*

Proof It suffices to prove

$$\sup_{g\in C_0^\infty([0,T]\times\mathbb{R}^6),\|g\|_{L^p([0,T];L^2)}=1}\int_0^T\int_{\mathbb{R}^6} f_2\partial_v g dtdxdv \le C. \tag{3.14}$$

Consider equation,

$$\partial_t\varphi + v\cdot\partial_x\varphi + E_{f_1}\cdot\partial_v\varphi - \beta v\cdot\partial_v\varphi + \sigma\Delta_v\varphi = \partial_v g. \tag{3.15}$$

From Remark 2.10, we can take φ in (3.15) as the test function in Definition 1.1.
We can prove the solution to (3.15) is unique as the proof of $\ker(\mathbb{L}^*) = \{0\}$. From the representation of solution via fundamental solution, we have

$$\begin{aligned}
&\|\varphi(0,x,v)\|_2\\
&=\left\|\int_T^0\int_{\mathbb{R}^6}\Gamma_1(x,v,0;x_0,v_0,\tau)\partial_{v_0}g(\tau,x_0,v_0)dx_0dv_0\right\|_2\\
&=\left\|-\int_T^0\int_{\mathbb{R}^6}\partial_{v_0}\Gamma_1(x,v,0;x_0,v_0,\tau)g(\tau,x_0,v_0)dx_0dv_0\right\|_2\\
&\le C\left\|\int_T^0\tau^{-7/10}\int_{\mathbb{R}^6}G(x/2,v/2,t;x_0/2,v_0/2)g(\tau,x_0,v_0)dx_0dv_0\right\|_2\\
&\le C\int_0^T\tau^{-7/10}\|g(\tau)\|_2\\
&\le C\|g(\tau)\|_{L^p([0,T];L^2)}\left\|\tau^{-7/10}\right\|_{L^{p'}([0,T])}.
\end{aligned}$$

Namely,

$$\|\varphi(0)\|_2 \le C.$$

Then from (1.2), we have proved (3.14), the proposition follows.

Lemma 3.4.

$$\|w(t)\|_{L^2}^2 = 0 \text{ if and only if } E_w = 0.$$

Proof From the Definition 1.1, for any test function φ, we know

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^6} w(\partial_t\varphi+v\cdot\partial_x\varphi + E_{f_1}\cdot\partial_v\varphi - \beta v\cdot\partial_v\varphi + \sigma\Delta_v\varphi)dtdxdv\\
&\qquad = -\int_0^T\int_{\mathbb{R}^6} f_2E_w\cdot\partial_v\varphi dxdvdt
\end{aligned} \tag{3.16}$$

Consider the following equation,

$$\partial_t\varphi + v\cdot\partial_x\varphi + E_{f_1}\cdot\partial_v\varphi - \beta v\cdot\partial_v\varphi + \sigma\Delta_v\varphi = w, \tag{3.17}$$

And from Proposition 2.7, there exists a sequence of Schwartz test functions φ_n, such that

$$\begin{aligned}
&\partial_t\varphi_n + v\cdot\partial_x\varphi_n + E_{f_1}\cdot\partial_v\varphi_n - \beta v\cdot\partial_v\varphi_n + \sigma\Delta_v\varphi_n \to w \text{ in } L^2;\\
&\qquad\qquad\qquad\qquad \varphi_n \to \varphi \text{ in } C([0,T];L^6).
\end{aligned}$$

Take φ_n as the test function in (3.16), then from

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^3} f_2E_w\cdot(\partial_v\varphi_n-\partial_v\varphi)dtdxdv\right|\\
&\leq\left|\int_0^T\int_{\mathbb{R}^3}\partial_v f_2\cdot E_w(\varphi_n-\varphi)dtdxdv\right|\\
&\leq\|\partial_v f_2\|_{L^{p'}([0,T];L^2)}\|E_w\|_{L^p([0,T];L^3)}\|\varphi_n-\varphi\|_{L^\infty([0,T];L^6)},
\end{aligned}$$

where $p>10/3$, and Proposition 3.3, let $n\to\infty$, we have

$$\|w\|_2\leq\left|\int_0^T\int_{\mathbb{R}^3} f_2E_w\cdot\partial_v\varphi dtdxdv\right|.$$

Thus the problem reduces to $E_w=0$.

4 Proof of theorem 1.3

Define $\Psi_m(x,v)=\Phi(x/m,v/m)$, and $\eta_m=\Psi_m(x,v)\int_{R^3}E(y)\frac{y-x}{|x-y|^3}dy$, then from Lemma 3.1, $\|\eta_m\|_\infty\leq C$. consider the following equation

$$\partial_t\varphi_m+v\cdot\partial_x\varphi_m+E_{f_1}\cdot\partial_x\varphi_m-\beta v\cdot\partial_x\varphi_m+\sigma\Delta_v\varphi_m=\eta_m\delta_{s,\theta}(t),$$

with solution

$$\varphi_m(t,x,v)=\int_T^t\Gamma_1(x,v,t;x_0,v_0,\tau)\delta_{s,\theta}(\tau)\eta_m(x_0,v_0)dx_0dv_0d\tau.$$

Similar to the proof of Proposition 3.3, by a limit argument, we can choose φ_m as the test function in Definition 1.1.
Define

$$\Omega(x,v,t,\tau)=\int_{\mathbb{R}^6}\partial_v\Gamma_1(x,v,t;x_0,v_0,\tau)\eta_m(x_0,v_0)dx_0dv_0,$$

then from (2.5), (2.6), Proposition 2.3, and bounded-ness of $\|\eta_m\|_\infty$, we obtain

$$\begin{aligned}
\|\Omega\|_{L^\infty(\mathbb{R}^6)}&\leq(t-\tau)^{-\frac{7}{10}}\int_{\mathbb{R}^6}G(\frac{x}{2},\frac{v}{2},t-\tau;\frac{x_0}{2},\frac{v_0}{2})\eta_m(x_0,v_0)dx_0dv_0\\
&\leq(t-\tau)^{-\frac{7}{10}}\|\eta_m\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^6}G(\frac{x}{2},\frac{v}{2},t-\tau;\frac{x_0}{2},\frac{v_0}{2})dx_0dv_0\\
&\leq C(t-\tau)^{-\frac{7}{10}} \qquad (4.18)
\end{aligned}$$

The L^2 estimate of $\rho(t)$, and (4.18), yield

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^6} f_2E_w\cdot\partial_v\varphi_m dtdxdv\right|\\
&\leq\left|\int_0^T dt\int_{\mathbb{R}^6} f_2E_w\cdot\int_T^t\int_{\mathbb{R}^6}\partial_v\Gamma_1(x,v,t;x_0,v_0,\tau)\delta_{s,\theta}(\tau)\eta_m(x_0,v_0)d\tau dx_0dv_0\right|\\
&\leq\left|\int_0^T\delta_{s,\theta}(\tau)d\tau\int_0^\tau dt\int_{\mathbb{R}^6} f_2E_w\cdot\Omega dxdv\right|
\end{aligned}$$

$$\leq \left| \int_0^T \delta_{s,\theta}(\tau)d\tau \int_0^\tau \|\Omega\|_\infty \|E_w\|_2 \left\| \int_{\mathbb{R}^3} f_2 \right\|_2 dt \right|$$
$$\leq \left| \int_0^T \delta_{s,\theta}(\tau)d\tau \int_0^\tau (t-\tau)^{-7/10} t^{-9/20} \|E_w(t)\|_2 dt \right|$$

Letting $\theta \to 0$, then $m \to \infty$, from Lemma 3.2, we deduce

$$\|E_w(s)\|_2 \leq C \int_0^s \|E_w(t)\|_2 t^{-9/20} (s-t)^{-7/10} dt.$$

We obtain from Henry type Gronwall inequality that,

$$E_w = 0.$$

Therefore Theorem 1.3 follows from Lemma 3.4.

5 Propagation of moments

Proposition 5.1. *If the initial data f_0 satisfies $f_0(1+|v|^k) \in L^1$, $f_0 \in L^\infty$, for some $k \geq 2$, then there exists a weak solution to VPFP system satisfying $f(t)(1+|v|^k) \in L^\infty([0,T]; L^1)$.*

Proof We split the proof into three steps.
Step1.
The existence of solution $f(t,x,v)$ is proved by F. Bouchut [2], since $k \geq 2$. The remaining part is to prove the propagation of moments. And again from F. Bouchut [2], the kinetic energy namely 2-order moment is preserved, and if denote the associated force field as E, then $\|E(t,x)\|_\infty \leq Ct^{-6/5}$.
Notice that it is proper to assume that we are dealing with smooth solutions, by a standard limit argument.
Denote the fundamental solution of

$$\partial_t f + v \cdot \partial_x f - \beta \mathrm{div}_v(vf) - \sigma \Delta_v f = 0,$$

as $H(x,v,t;x_0,v_0)$, and in fact

$$\begin{aligned} H(x,v,t;x_0,v_0) = \frac{1}{(2\pi)^6} \left(\frac{\pi}{\sigma\sqrt{D}} \right)^3 \exp\{ & \frac{-1}{4\sigma D} [\frac{1}{2\beta}(1-e^{-2\beta t})|\bar{x}|^2 - \\ & \left(\frac{2}{\beta^2}(1-e^{-\beta t}) - \frac{2}{\beta^2}(1-e^{-2\beta t}) \right) \langle \bar{x}, \bar{v} \rangle + \\ & \left(\frac{t}{\beta^2} - \frac{2}{\beta^2}(1-e^{-\beta t}) + \frac{2}{2\beta^3}(1-e^{-2\beta t}) \right) |\bar{v}|^2] \}, \end{aligned}$$

where $\bar{x} = x - (x_0 + \frac{v_0}{\beta}(1-e^{-\beta t}))$, $\bar{v} = v - v_0 e^{-\beta t}$, and $D = \frac{\beta t(1-e^{-2\beta t}) - 2(1-e^{-\beta t})^2}{2\beta^4}$.
Step2.
Then by the same arguments in Proposition 2.3, we can prove the existence of fundamental solution Γ_3 to

$$\partial_t f + v \cdot \partial_x f + E \cdot \partial_v f - \beta \mathrm{div}_v(vf) - \sigma \Delta_v f = 0,$$

and Γ_3 satisfies

$$|\Gamma_3(x,v,t;x_0,v_0,\tau)| \leq C(t-\tau)^{-\frac{1}{5}} H((x,v,t-\tau;x_0,v_0).$$

The solution $f(t,x,v)$ can be expressed by

$$f(t,x,v)=\int_{\mathbb{R}^6}\Gamma_3(x,v,t;x_0,v_0,0)f_0(x_0,v_0)dx_0dv_0.$$

Then, it is easy to see

$$\begin{aligned}
&\int_{\mathbb{R}^6}|v|^k f(t,x,v)dxdv\\
&=\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\Gamma_3(x,v,t;x_0,v_0,0)|v|^k f_0(x_0,v_0)dx_0dv_0dxdv\\
&\le Ct^{-1/5}\int_{\mathbb{R}^6}\int_{|v|\ge 2|v_0|e^{T|\beta|}}H(x,v,t;x_0,v_0)|v|^k f_0(x_0,v_0)dx_0dv_0dxdv\\
&\quad+Ct^{-1/5}\int_{\mathbb{R}^6}\int_{|v|\le 2|v_0|e^{T|\beta|}}H(x,v,t;x_0,v_0)|v|^k f_0(x_0,v_0)dx_0dv_0dxdv\\
&\le Ct^{3/10}\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}H(x/2,v/2,t;x_0/2,v_0/2)f_0(x_0,v_0)dx_0dv_0dxdv\\
&\quad+Ct^{-1/5}\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}H(x,v,t;x_0,v_0)|v_0|^k f_0(x_0,v_0)dx_0dv_0dxdv\\
&\le Ct^{-1/5}\left\|f_0(1+|v|^k)\right\|_{L^1},
\end{aligned}\tag{5.19}$$

where we have used $|\bar{v}|\,H(x,v,t;x_0,v_0)\le Ct^{1/2}H(x/2,v/2,t;x_0/2,v_0/2)$. and $2|\bar{v}|\ge|v|$, when $|v|\ge 2|v_0|\,e^{T|\beta|}$.

Step3.

In addition, from the proof of Lemma 4.3 in [7], we have

$$\begin{aligned}
&\partial_t\int_{\mathbb{R}^6}f(t,x,v)\langle v\rangle^k dxdv\\
&=-\int_{\mathbb{R}^6}(v\cdot\partial_x f+\operatorname{div}_v(E-\beta v)f-\sigma\Delta_v f)\langle v\rangle^k dxdv\\
&\le C\int_{\mathbb{R}^6}|E|\,f\langle v\rangle^{k-1}dxdv+\beta\langle v\rangle^k f+\sigma f\langle v\rangle^{k-2}dxdv\\
&\le C\|E\|_{3+k}\left\|f\langle v\rangle^k\right\|_1^{\frac{k+2}{k+3}}+C\left\|\langle v\rangle^k f\right\|_1+C\left\|f\langle v\rangle^{k-2}\right\|_1
\end{aligned}$$

After integrating the above formula in $[0,t]$, it follows

$$\begin{aligned}
\left\|\langle v\rangle^k f(t)\right\|_1\le\left\|\langle v\rangle^k f_0\right\|_1&+\int_0^t\|E(s)\|_{3+k}\left\|f(s)\langle v\rangle^k\right\|_1^{\frac{k+2}{k+3}}+\left\|\langle v\rangle^k f(s)\right\|_1 ds\\
&+\int_0^t C\left\|f(s)\langle v\rangle^{k-2}\right\|_1 ds.
\end{aligned}$$

From [2], $\|E\|_{3+k}$ is integrable in $[0,t]$, when $k\le 6$, in fact $\|E\|_r\le Ct^{-6/5+9/2r}$, and from (5.19), $\left\|f(s)\langle v\rangle^{k-2}\right\|_1$ is also integrable in $[0,t]$, thus

$$\left\|\langle v\rangle^k f(t)\right\|_1\le C,$$

namely the moment propagates.

Appendix A. The proof of $\eta(0)=0$ in Proposition 2.9

Lemma A.1 *For any $(n,m)\in\mathbb{N}\times\mathbb{N}$, there exists $C(n,m)$, such that for $N\geq 1$,*

$$\left\|P_{n,m}\langle v\rangle^{-N}g\right\|_Y \leq C(n,m)\|g\|_2.$$

Proof

We take $\left\|(1+|v|)\Delta_v P_{n,m}\langle v\rangle^{-N}g\right\|_2 \leq C(n,m)\|g\|_2$ as an example, the others are similar. We derive from Bernstein's inequality that,

$$\begin{aligned}
&\left\|(1+|v|)\Delta_v P_{n,m}\langle v\rangle^{-N}g\right\|_2\\
&\leq C\left\|\Delta_v[\langle v\rangle\, P_{n,m}\langle v\rangle^{-N}g]\right\|_2 + C\left\|\partial_v P_{n,m}\langle v\rangle^{-N}g\right\|_2 + C\left\|P_{n,m}\langle v\rangle^{-N}g\right\|_2\\
&\leq C\left\|\Delta_v F_{\xi,\zeta}{}^{-1}\left\langle\nabla_\zeta\right\rangle \Psi F_{x,v}\langle v\rangle^{-N}g\right\|_2 + C2^m\left\|P_{n,m}\langle v\rangle^{-N}g\right\|_2\\
&\leq C\left\|\Delta_v F_{\xi,\zeta}{}^{-1}\left\langle\nabla_\zeta\right\rangle [\Psi F_{x,v}\langle v\rangle^{-N}g]\right\|_2 + C2^m\|g\|_2\\
&\leq C\left\|\Delta_v F_{\xi,\zeta}{}^{-1}\left(\left\langle\nabla_\zeta\right\rangle \Psi\right)\left(F_{x,v}\langle v\rangle^{-N}g\right)\right\|_2 + C\left\|\Delta_v F_{\xi,\zeta}{}^{-1}\Psi\left\langle\nabla_\zeta\right\rangle F_{x,v}\langle v\rangle^{-N}g\right\|_2 + C2^m\|g\|_2\\
&\leq C2^m\|g\|_2 + C\left\|\Delta_v F_{\xi,\zeta}{}^{-1}\Psi F_{x,v}\langle v\rangle^{-N+1}g\right\|_2 + C2^m\|g\|_2\\
&\leq C2^{2m}\|g\|_2.
\end{aligned}$$

Proof of $\eta(0)=0$ in Proposition 2.9.

Before going to prove the claim in Proposition 2.9, we point out the following fact: from fundamental theorem of calculus, for $\tilde{\varphi}$ in Schwartz class, in Y' it holds that,

$$\int_o^T \partial_t\tilde{\varphi}P_{n,m}\eta(t)dt = -\int_o^T \tilde{\varphi}\partial_t P_{n,m}\eta(t)dt - P_{n,m}\eta(0)\tilde{\varphi}(0),$$

the integral are regarded in Bochner's sense. We mention $P_{n,m}\eta(0)\tilde{\varphi}(0)\in L^1(\mathbb{R}^6)$, in fact we have $P_{n,m}\eta(0)\langle v\rangle^{-N}\in L^2$, for N sufficiently large, indeed

$$\begin{aligned}
\left\|P_{n,m}\eta(0)\langle v\rangle^{-N}\right\|_2 &= \sup_{\|g\|_2=1}\int_{\mathbb{R}^6} P_{n,m}\eta(0)\langle v\rangle^{-N}g(x,v)dxdv\\
&\leq \sup_{\|g\|_2=1}\int_{\mathbb{R}^6}\eta(0)P_{n,m}\langle v\rangle^{-N}gdxdv\\
&\leq \sup_{\|g\|_2=1}\|\eta(0)\|_{Y'}\left\|P_{n,m}\langle v\rangle^{-N}g\right\|_Y\\
&\leq \|\eta(0)\|_{Y'}C(n,m),
\end{aligned}$$

where we have used Lemma A.1. And similarly $\int_o^T \partial_t\tilde{\varphi}P_{n,m}\eta(t)dt\in L^1(\mathbb{R}^6)$, then $-\int_o^T\tilde{\varphi}\partial_t P_{n,m}\eta(t)dt\in L^1(\mathbb{R}^6)$, and

$$\int_{\mathbb{R}^6\times[0,T]}\partial_t\tilde{\varphi}P_{n,m}\eta(t) = -\int_{\mathbb{R}^6\times[0,T]}\tilde{\varphi}\partial_t P_{n,m}\eta(t) - \int_{\mathbb{R}^6}P_{n,m}\eta(0)\tilde{\varphi}(0). \tag{5.20}$$

Now we turn to prove $\eta(0)=0$. From the definition of adjoint operator, we have for any $\varphi\in D(L)$, any $\eta\in D(\mathbb{L}^*)$we have

$$\int_{[0,T]\times\mathbb{R}^6}(\partial_t\varphi + v\cdot\partial_x\varphi + E\cdot\partial_x\varphi - \beta v\cdot\partial_v\varphi + \sigma\Delta_v\varphi)\eta dxdvdt \tag{5.21}$$

$$= \int_{[0,T]\times\mathbb{R}^6} (-\partial_t \eta - v\cdot\partial_x\eta - E\cdot\partial_x\eta + \beta v\cdot\partial_v\eta + \sigma\Delta_v\eta)\varphi dxdvdt \tag{5.22}$$

Take $\varphi = P_{n,m}\tilde{\varphi}$, $\tilde{\varphi}(t,x,v) = w(t)h(x)r(v)$, where $h, r \in S$, $w \in C^\infty$, and $w(T) = 0$.
For (5.21), direct calculations and the self-adjoint-ness of $P_{n,m}$ implies

$$\begin{aligned}
&\int_{[0,T]\times\mathbb{R}^6} (\partial_t P_{n,m}\tilde{\varphi} + v\cdot\partial_x P_{n,m}\tilde{\varphi} + E\cdot\partial_x P_{n,m}\tilde{\varphi} \\
&\qquad - \beta v\cdot\partial_v P_{n,m}\tilde{\varphi} + \sigma\Delta_v P_{n,m}\tilde{\varphi})\eta \\
&\quad = \int_{[0,T]\times\mathbb{R}^6} (\partial_t\tilde{\varphi} + v\cdot\partial_x\tilde{\varphi} + E\cdot\partial_x\tilde{\varphi} - \beta v\cdot\partial_v\tilde{\varphi} + \sigma\Delta_v\tilde{\varphi})P_{n,m}\eta \\
&\qquad + \int_{[0,T]\times\mathbb{R}^6} (-\partial_x F^{-1}\partial_\zeta\Psi F\tilde{\varphi} + \beta\partial_v F^{-1}\partial_\zeta\Psi F\tilde{\varphi})\eta \\
&\qquad + \int_{[0,T]\times\mathbb{R}^6} (-P_{n,m}E\cdot\partial_v\tilde{\varphi} + E\cdot\partial_v P_{n,m}\tilde{\varphi})\eta \\
&\quad \triangleq A_1 + B_1 + B_2
\end{aligned}$$

And from (5.20) and integration by parts, direct calculations indicate,

$$\begin{aligned}
A_1 = &- \int_{\mathbb{R}^6} \tilde{\varphi}(0)P_{n,m}\eta(0)dxdv \\
&+ \int_{[0,T]\times\mathbb{R}^6} (-\partial_t P_{n,m}\eta - v\cdot\partial_x P_{n,m}\eta - E_{f_1}\cdot\partial_x P_{n,m}\eta \\
&+ \beta v\cdot\partial_v P_{n,m}\eta + \sigma\Delta_v P_{n,m}\eta)\tilde{\varphi} \\
= &\int_{[0,T]\times\mathbb{R}^6} (-\partial_t\eta - v\cdot\partial_x\eta - E\cdot\partial_x\eta + \beta v\cdot\partial_v\eta + \sigma\Delta_v\eta)P_{n,m}\tilde{\varphi} \\
&+ \int_{[0,T]\times\mathbb{R}^6} (\partial_x F^{-1}\partial_\zeta\Psi F\eta - \beta\partial_v F^{-1}\partial_\zeta\Psi F\eta)\tilde{\varphi} \\
&+ \int_{[0,T]\times\mathbb{R}^6} (P_{n,m}E\cdot\partial_v\eta - E\cdot\partial_v P_{n,m}\eta)\tilde{\varphi} \\
&- \int_{\mathbb{R}^6} \tilde{\varphi}(0)P_{n,m}\eta(0)dxdv \\
\triangleq &A_2 + B_3 + B_4 - \int_{R^6} \tilde{\varphi}(0)P_{n,m}\eta(0)dxdv,
\end{aligned}$$

By Bernstern's inequality, we have

$$\lim_{m,n\to\infty} |B_1| + |B_3| = 0.$$

Indeed

$$\begin{aligned}
|B_1| &\le \int_0^T \frac{C}{2^n}\|\eta\|_2\|\partial_x\tilde{\varphi}\|_2 dt + \int_0^T \frac{C}{2^m}\|\eta\|_2\|\partial_v\tilde{\varphi}\|_2 dt \\
&\le \frac{C}{2^n}\|\partial_x\tilde{\varphi}\|_{L^2([0,T];L^2)}\|\eta\|_{L^2([0,T];L^2)} + \frac{C}{2^m}\|\partial_x\tilde{\varphi}\|_{L^2([0,T];L^2)}\|\eta\|_{L^2([0,T];L^2)},
\end{aligned}$$

B_3 is the same.
And we claim

$$\lim_{n,m\to\infty} |B_2| + |B_4| = 0.$$

In fact, B_2 term follows from

$$
\begin{aligned}
|B_2| &\le \int_0^T \|E\|_2\|\eta\|_2 \left\|\int_{\mathbb{R}^3} |P_{n,m}\partial_v\tilde{\varphi} - \partial_v\tilde{\varphi}|\, dv\right\|_\infty dt + \int_0^T \|E\|_2 \left\|\int_{\mathbb{R}^3} |\partial_v\tilde{\varphi}|\, dv\right\|_\infty \|P_{n,m}\eta - \eta\|_2 dt \\
&\le \left\|\int_{\mathbb{R}^3} |P_{n,m}\partial_v\tilde{\varphi} - \partial_v\tilde{\varphi}|\, dv\right\|_{L^\infty([0,T];L^\infty)} \|\eta\|_{L^2([0,T];L^2)}\|E\|_{L^2([0,T];L^2)} \\
&\quad + \|P_{n,m}\eta - \eta\|_{L^2([0,T];L^2)} \left\|\int_{\mathbb{R}^3} |\partial_v\tilde{\varphi}|\, dv\right\|_{L^\infty([0,T];L^\infty)} \|E\|_{L^2([0,T];L^2)}.
\end{aligned}
$$

The proof of B_4 is the same.
Then combining (5.21)=(5.22), (5.21)= A_1, (5.22)= A_2, we deduce that

$$
\lim_{n\to\infty}\lim_{m\to\infty}\int_{\mathbb{R}^6} \tilde{\varphi}(0)P_{n,m}\eta(0)dxdv = 0.
$$

for any smooth $\tilde{\varphi}(0)$, therefore, we have proved $\lim\limits_{n\to\infty}\lim\limits_{m\to\infty} P_{n,m}\eta(0) \to 0$, in distribution sense, but $P_{n,m}\eta(0) \to \eta(0)$ in distribution, thus $\eta(0) = 0$.

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